Bow and Arrow Efficiency

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Introduction

People have been shooting bows and arrows at animals, targets and each other for over 5,000 years. Even the earliest bows were marvels of design efficiency, taking advantage of the unique structural properties of wood. What better way was there of utilizing wood, a material that can only be compressed or stretched a mere 1 %, for propelling a projectile at over 49 meters/second (160 ft/sec)? An interesting and underestimated feature is that a bow transfers, with over 60 % efficiency, human effort to potential energy in the bow limbs weighing on the order of .36 kilograms (0.8 pounds) to kinetic energy of an arrow weighing .0226 kilogram (350 grains), a mass ratio of 16 to 1. This remarkable efficiency is due to efficient leverage. Just before the arrow separates from the bowstring a very small movement of the bow limbs causes a large movement of the arrow.

Objectives:

The objectives of this paper are:

1. To present an accurate but tractable physical model for the performance of a straight-limbed wooden bow. Contemporary archers have available bows with exotic designs made from space-age materials but there has always been a desire to gain a deeper understanding of how our early ancestors did things.
2. To compare the results of detailed computer simulation with the simpler model presented here. Other investigators have applied very sophisticated computer analytical tools to the analysis of bow and arrow dynamics. This simpler model presented here is worthwhile only if it gives similar accurate results.
3. To compare modeling results with experimental performance of straight-limbed wood bows. The immediate conclusion drawn from a comparison between actual performance and the computer modeling based solely on the elastic modulus and density of the bow limb material is that internal friction (hysteresis) in the bow limbs in a major contributor to inefficiency in a wooden bow. Internal friction is also very difficult to characterize.

Requirements of the model:

1. That it accurately duplicate the force-draw curve. The area under the force-draw curve tells us how much potential energy is available. The efficiency of the system is the arrow's kinetic energy divided my the available potential energy.
2. That it accurately predict the arrow speed versus arrow mass. Different archers have different needs. The bow hunter wants a heavier arrow to maximize efficiency and penetration. The flight shooter wants to shoot as far as possible and consequently uses a very light arrow. The generally agreed-upon rule for comparing the efficiency of bows with different draw weights is to measure arrow speed with an arrow mass of 10 grains per pound force (lbf) of draw weight, all at a draw length of 28 inches (71.12 cm). 1 lbf = 4.4484 newtons, 7000 grains = 1 lb = .4536 kg, 1 inch = 2.54 cm.
3. That it be mathematically tractable. Shareware is available that can automatically solve for the dynamics of complex mechanical structures. They lose insight. The model presented here reduces the mechanical structure of each bow limb to a rigid handle section, two massless limb sections connected by hinged joints with torsion springs and one point mass.

The model should include the shape, density, elastic modulus and internal friction in the bow limbs and the mass and elasticity of the bowstring. Density and elastic modulus are known and easily measured but internal friction or hysteresis in the bow limbs is not.

Previous work:

In 1937 Hickman presented a simple model for a straight limbed bow with stretchless bowstring whose limbs bend in circular arcs.(1) It consisted of a massless lever arm, ¾ as long as the working part of the limb with a point mass concentrated at the tip and a torsion spring connecting the lever arm to the rigid handle section. One third of the string mass was added to the arrow mass since the center of the bowstring moved at the same speed as the arrow and the ends of the bowstring had very little speed just before the arrow is released. This model gave remarkably accurate results for the force-draw curve of a straight-limbed bow but was an oversimplification of the dynamic properties because it claimed that the string tension would go to infinity just before the arrow was released and it predicted that all the stored energy in the static force-draw curve would be transferred to kinetic energy of the arrow and bowstring. This was only the beginning. It was exceptional in that it accomplished so much before computers were developed.

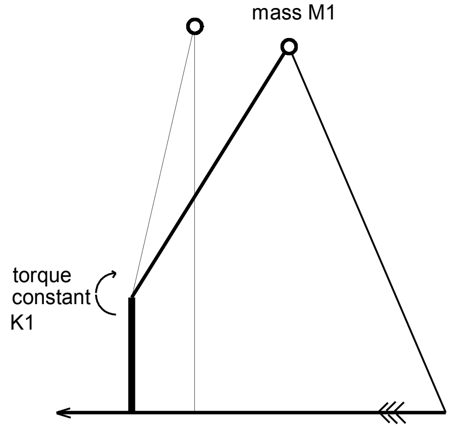


Fig. 1. Hickman's bow model.

Marlow attacked a serious weakness in Hickman's model by including the elasticity of the bowstring.(2) Today that is probably a moot point because modern bowstrings have so little stretch. Limb inertia and vibration were not addressed since he used same model for the bow limb as used by Hickman.

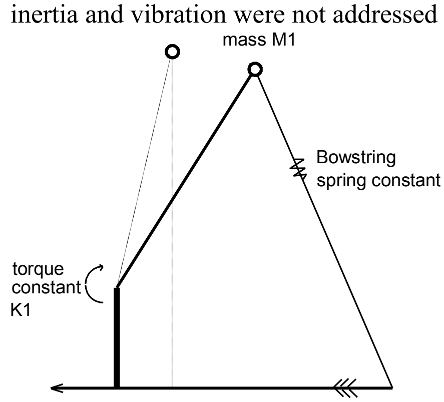


Fig. 2. Marlow's bow model.

For his PhD thesis Kooi developed a very detailed computer-based model that included everything except internal friction in the bow limbs.(3) The bow limb was modeled as a large number of point masses separated by torsion springs. The point masses and torsion springs were selected to match the mass density and stiffness along the bow limb. Because of the varying angles between the different elements the equations of motion are soluble only with very complex mathematical routines.

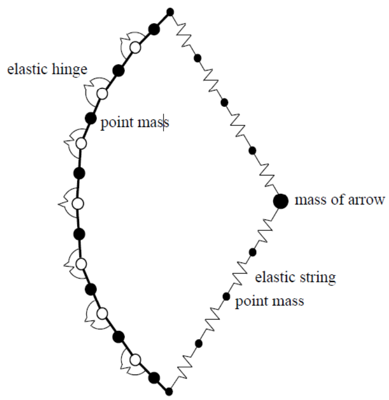


Fig. 4. Kooi's bow model.

Simplified model

In the Simplified model of Fig. 5 the upper and lower limbs of a straight-limbed bow are each represented by a rigid, non-moving handle section, an inner limb and an outer limb. The three are connected via two torsion springs with torque constants K1 & K2. The inertia of the bow limbs is represented by a point mass, M2, on the end of the inner limb. Elasticity of the bowstring can be included in this model but is not in this paper. The lengths of the sections, the point mass and the values of the torsion springs are chosen to match the force-draw curve, tip trajectory and the response to a horizontal step force applied to the tip. Replacing a complicated mechanical structure with a simpler one with similar dynamics is analogous to the problem in electrical engineering of representing a complicated electrical network with a simplified 'equivalent circuit'.

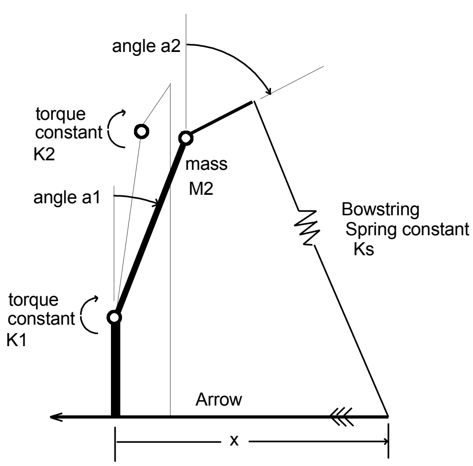


Fig. 5. The Simple bow model (upper limb only).

Comparison with Kooi's model

It is useful to compare the results of this Simple model of the KL bow and arrow with the numbers obtained from Kooi's much more detailed model. The characteristics to be compared are the drawing force at 28 inches, energy stored in the bow at full draw, the arrow speed for a specified arrow and bowstring mass.

Model	Draw wt @ 28 in	Stored Energy, ft-lb	Arrow Speed, ft/sec	Efficiency
Kooi	35.08 lbf	33.31	172.65	.765
Simple	34.9 lbf	33.86	176.1	.78

Comparison with experimental data

TBB4 (6) includes plenty of experimental data on arrow speed under standardized conditions with wood bows. In order to compare the efficiency of bows with different draw weights arrow speed is measured with an arrow mass of ten grains per pound of draw force measured at 28 inches. Under those conditions the KL bow model yields an arrow speed of 183.6 fps. The best arrow speed under those conditions for a 'real' wood bow made by an expert bowyer is 164 fps. This difference is even more dramatic when we consider the fact that the KL bow is not highly strained (0.83 %) whereas the 'real' bow was tuned for highest performance. Less talented bowyers achieve an arrow speed as low as 150 fps. The conclusion one must make is that if one includes only the density and elastic modulus of the bow limbs then the computation of arrow speed and bow efficiency is seriously in error. The neglected factor is hysteresis or internal friction in the bow limbs. Hysteresis is notoriously difficult to measure because it is time-dependent and highly dependent on the amount of strain in the wood. It must be measured under exactly the same conditions used in a bow. It is common knowledge among wooden bow aficionados that to obtain maximum arrow speed the bow must be drawn and released as quickly as possible. One could say that the best way to measure the hysteresis in bow limbs is to compute the arrow speed based on the dimensions, elastic modulus and density of the wood and then experimentally measure the arrow speed. The difference is due to hysteresis.

Bowyers are always asking "What can I do to make my bows faster?" The answer is to obtain a better understanding of hysteresis and how to minimize it. How does it depend on the amount of strain in the wood? Can it be modified by heat treatment of the wood? How does it vary with the time held at full draw?

What else can be learned from computer modeling?

Having a fairly accurate bow and arrow dynamics calculator makes it possible to compare different bow designs without having to actually build the bows. It also gives the ability to dispel some of the bow builders' folklore. First, one must keep in mind what is a fair comparison? It would not be fair to compare one bow with very highly stressed limbs with another with lower limb stress. I think it is fair to say that the comparison should be made between bows that have the same brace height, draw weight, full draw length and maximum strain in the limbs. What about reflex or its opposite, string follow? Be alert!

What about the old adage lifted from an internet bowyer's forum that "Short bows shoot faster than longbows?" We know that the shorter bow will store less energy in its force-draw curve but some claim that the short bow responds more "quickly". Truth or fantasy? Comparing three different length straight-limbed 40# at 28 inch yew bows with trapezoidal width taper from riser to tips, .5 inch tips, thickness adjusted for circular arc tillering and similar limb strain shooting 400 grain arrows we have:

Length, In	Width at Riser, in	Thickness at Riser, in	Riser length, In	strain	Stored energy, ft-lb	Arrow Speed, ft/sec
72	1.5	.635	8	.0085	35.9	178.5
60	2.5	.418	8	.0088	36.1	178.3
54	3.0	.335	8	.0093	34.1	175.6

This says that for similar strain level the 54 inch bow doesn't shoot as fast as the 60 inch or 72 inch bows.

In doing science experiments it is often valuable (and difficult) to change just one variable and observe the difference in performance. One would like to make several different bows out of the same piece of wood and see which performs best. That's very difficult because of the variability in wood. We can, however, do that in a computer simulation. For example, making bow limbs proportionally wider, for the same draw weight, will reduce the strain. How much can the strain be reduced and how much arrow speed is sacrificed by making bow limbs wider? The reduction is strain is easy to calculate because the stiffness of the limb is proportional to the width multiplied by the cube of the thickness. If you make the limbs twice as wide and reduce the thickness to achieve the same draw weight then the thickness and strain are reduced to 79% of their original value. As an example, several hickory bows with 0.5 inch tips, rectangular cross section limbs whose thicknesses were adjusted to give circular arc bending, 40 lb @ 28 in, 200 grain bowstring and 400 grain arrows were modeled. The results:

Width at riser, inches	Thickness at riser, Inches	Arrow speed, Ft/sec	Strain (relative)
1.5	.503	181.0	1.0
2.0	.457	178.3	.909
2.5	.416	176.4	.827

We see that increasing the limb width from 1.5 in (typical) to 2.5 in (extremely wide) reduces the strain by about 17 % but also reduces arrow speed by only 2.5 %. The question that the modeling cannot answer is whether or not that reduced strain also reduces hysteresis loss by any significant amount.

A familiar mantra of the wooden bow makers is "Keep the tips narrow for greatest arrow speed." Is that what really matters or is that just a way to force the bowyers to make their limbs bend in circular arcs? Even for circular arc tillering there will be more mass distributed over the outer part of the limbs for a bow with wider tips. How big is that effect? We compute the arrow speed for three bows, similar to the KL bow discussed earlier but with trapezoidal width, 1.5 inches wide at the riser and 0.25, 0.50 and 0.75 inches wide at the tips, thickness adjusted to give 40 lb at 28 inches and circular arc bending, a 400 grain arrow and 105 grain bowstring. The conclusion is that as long as you tiller for circular arc the wider tips make practically no change in arrow speed. I would claim that the degradation of arrow speed observed with wide-tipped bows is because the bowyer is NOT tillering for circular arc. Instead the bows have stiff and consequently heavier outer limbs. Another conjecture is that these bows put more strain on the part of the limb nearest to the riser and the additional strain causes more hysteresis loss.

Tip width, inches	Arrow speed, ft/sec Circular arc tiller	Arrow speed, ft/sec Constant thickness
.25	181.6	184.3
.50	178.6	173.6
.75	179.6	164.3

We see that the worst performer has .75 inch (wide) tips and constant thickness, giving 'stiff' ends. As long as the thickness is adjusted to give circular arc bending over the entire length of the limb there is not much difference in arrow speed. Only when you have wide, heavy tips is the arrow speed impaired.

It is interesting to compare arrow speed achieved with a stick bow with what can be achieved with bow limbs made from modern materials. A 50.5 pound bow with fiberglass limbs reflexed one or two inches shoots 457 grain arrows at 190 ft/sec with a mechanical release and 185 ft/sec with a three-finger (human) release. A similar 53 pound bow with carbon fiber limbs shoots 540 grain arrows at 192.2 ft/sec with a mechanical release. The improved performance over wood bows is mainly due to greatly reduced hysteresis.

References:

1 C.N. Hickman, "Dynamics of a Bow and Arrow" Journal of Applied Physics, June, 1937. Reproduced in "Archery-the technical Side", pp 69-76.

2 W.C. Marlow, "Bow and Arrow Dynamics", Am. J. Phys. 49(4), April 1981.

3 B.W. Kooi, "On the Mechanics of the Bow and Arrow", Thesis, Rijksuniverseit Groningen, 1983. Available at http://www.bio.vu.nl/thb/users/kooi/

4 B.W. Kooi, "The 'Cut and Try' Method in the Design of the Bow", Available at his website, http://www.bio.vu.nl/thb/users/kooi/

5. P.E Klopsteg, "Constructing the Bow with Rectangular Limb Section", Archery -the Technical Side

6. Traditional Bowyer's Bible, Volume 4, page110.

Appendix: Details of the model

From dimensions to model

The bow to be modeled, the 'KL' bow of Fig. 6 was proposed by Klopsteg in 1932 (5). His objective was to design the bow limbs to give maximum transfer of stored energy to the arrow consistent with the greatest safety factor. In order to insure that no part of the limb was strained more than another rectangular cross-section limbs of constant thickness bending in circular arcs were specified. That required limb width to vary linearly from handle to tip:

w_ideal(s) = W0*(1- s/Llimb) 0<s< Llimb

This was physically unrealizable because it resulted in zero tip width. The compromise made for the KL bow was to have the limb width taper uniformly: w(s) = W0*(1- s/L1) for 0<s< 2/3*L1 and = Wt for 2/3*L1<s<L1.

Where 'Ll' is the length of the working part of limb from rigid handle to tip.

Other dimensions of the KL bow:

Overall length = 72 inches

Rigid handle section = 8 ''

Limb width at handle = W0 = 1.5''

Limb width at tip = Wt = 0,5"

Limb thickness = Tl = 0.60 ''

It is made from yew wood with elastic modulus (E)= 1.46 million psi and density (D) = 43 lb/ft^3. Initially any internal friction or hysteresis in the bow limbs is ignored. This particular bow was modeled for two reasons. It is a "good" design and it has already been extensively modeled in much greater detail by Kooi (4). The results described here are for a specific bow but the technique is applicable to bows of arbitrary dimensions.

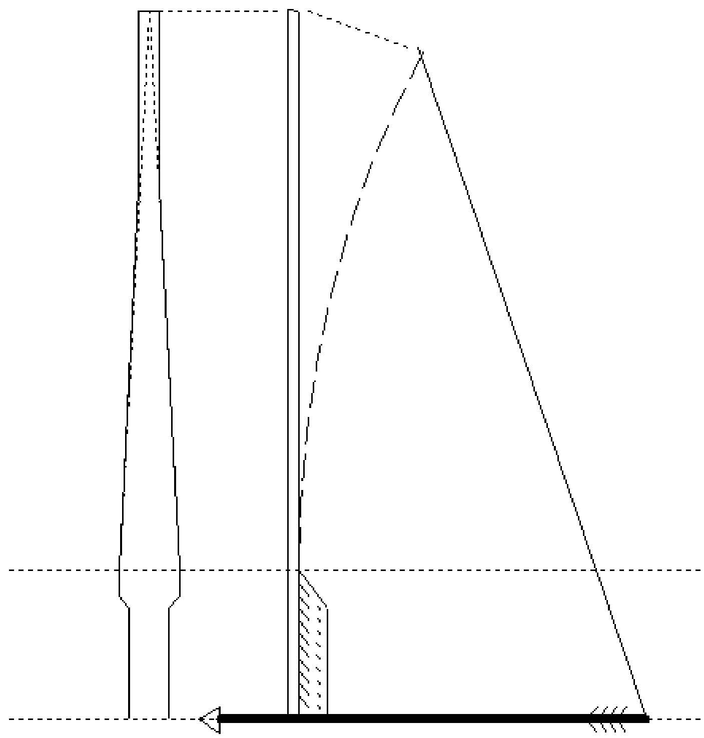


Fig. 6. The Klopsteg (KL) bow.

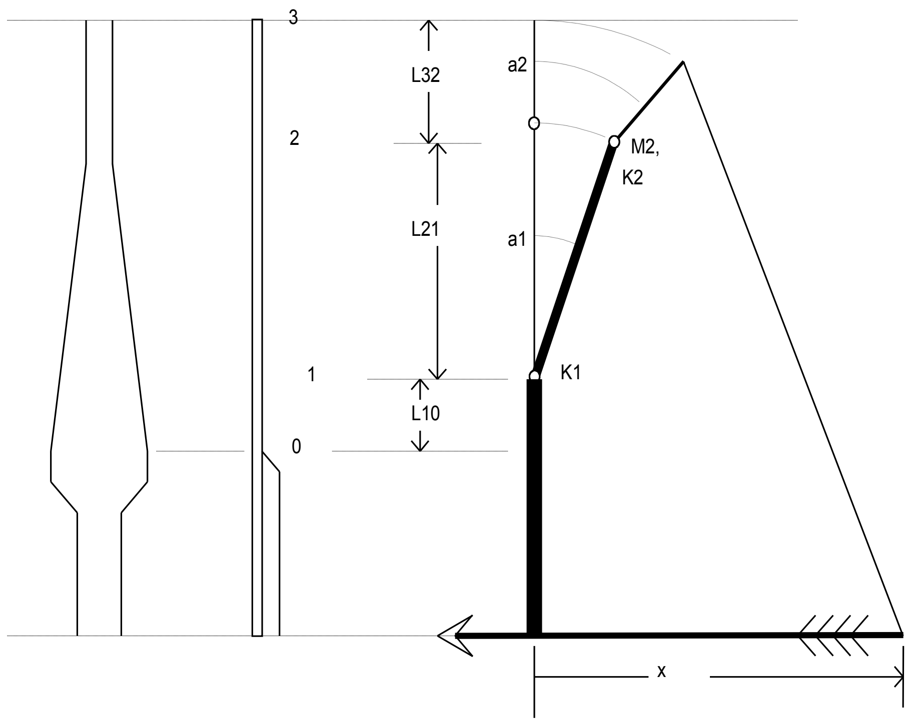


Fig. 7. Modeling parameters for the KL bow.

From a continuous to lumped model

Much of the difficulty in doing an accurate analysis of bow dynamics arises because of the non-perpendicularity of the forces. The relative angles between the bowstring and elements of the bow limbs vary with time. One can, however, obtain an excellent dynamic match between the simple model and a more detailed model, (Fig. 7) by matching their responses to a small perpendicular force applied to the tip. The procedure is:

1. Compute the stiffness and density as functions of position along the limb.
2. Apply a small steady-state perpendicular force to the tip of the detailed model and calculate the displacement along the limb.
3. Arbitrarily select a location for the second joint.
4. Determine, based on the location of the second joint and the displacement along the limb, the location of the first joint and the two torque constants.
5. Compare the response of the Simple model to a small perpendicular step force applied to the tip of this structure with the response of a more detailed (six joint) model. This is straightforward because there are negligible departures from perpendicularity.
6. Vary the location of the second joint and the mass,M2, to match the step response of the more detailed

structure of Fig. 8.

First the stiffness, EI(s), and mass per unit length ,Rho(s), along the limb are computed.
EI(s) = (elastic modulus)*(cross-sectional moment of inertia) = 1/12 * E*w(s)*Tl^3
and Rho(s) = D*Tl*w(s) for the 0<s<L1, the length of the working limb.

Next, x(s) and y(s), the horizontal and vertical displacement of each part of the limb due to a small steady-state force, *Force*, applied to the tip are computed.
Second derivative =
x2(s) = *Force*lever arm)*/stiffnes = *Force**(L1-s)/EI(s) = second derivative

$$x1(s) = \int_0^s x2(u)du$$
= slope (first derivative)

$$x(s) = \int_0^s x1(u)du$$
= horizontal displacement

$$y(s) = \int_0^s \sqrt{1 - (x(u))\hat{}2}du$$
= vertical displacement

x(s) and y(s) plus the location of joint #2 will be used to determine the two torque constants and the locations of joint #1.
Given Y2 = y(L20), X2 = x(L20)
and
Y2= L10 + sqrt[L21^2 -X2^2]
One can solve exactly
L21 =[(Y2-L20)^2 + X2^2]/[2*(L20-Y2)].

Given L20 and the relationship
Xt(L20 = Force*Lx/K1, xt(…,
We derive
L21=½*xt(L20)/(L20-y(L20)
L10=L20-L21
L32=(1-f)*Llimb =L1-L20
L31=L21+L32
K1=Force*L31*L21/xt(L20)

K2= [Force*L32^2]/[x(L30)-Force*L31^2/K1]

L20 and the mass M2 are chosen to give the best dynamic match to the tip motion from a small horizontal step force applied to the tip. The potential energy of the six-joint model is
V6(x1, x2,…x6)=½*[k0*x1^2 + k1*(x2-2*x1+0)^2 + k2*(x3-2*x2+x1)^2 + k3*(x4-2*x3+x2)^2 + k4*(x5-2*x4+x3) + k5*(x6-2*x5+x4)^2]
And kinetic energy
T(v1, v2...v6) = ½*[m1*v1^2 + m2*v2^2 +…m6^v6^2]
Deriving the equations of motion from the expressions for potential and kinetic energy is relatively straightforward.

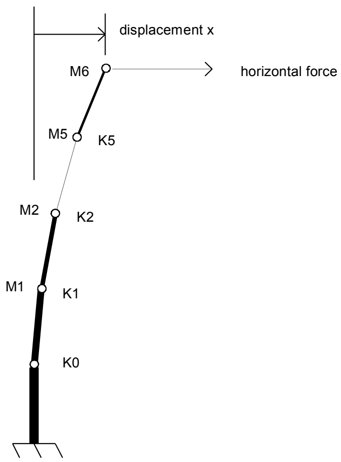


Fig. 8. A six-joint model for the upper bow limb.

The calculations for the Simplified model are done with Mathcad, a general mathematics software package using the Lagrangian formulation for the system dynamics. The Lagrangian (kinetic energy - potential energy) using the position of the center of the bowstring and the angle of the inner bow limb (a1) as the two independent coordinates. The angle between the vertical and the upper segment of the bow limb, a2 is a derived quantity, a function of a1 and x, a2(a1,x). The potential energy, assuming initially straight bow limbs before attaching the bowstring, is

V(x, a1, a2) = ½ * [K1*a1^2 + K2*(a2-a1)^2]

The kinetic energy is

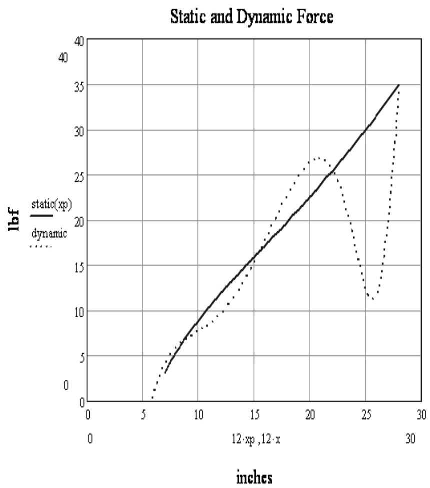
T(va1, vx) = ½ * (M2*L1^2*va1^2 + Mas*vx^2)

Where va1 is the angular velocity of the angle a1, vx is the velocity of the arrow and Mas is the mass of the arrow plus one third of the bowstring. One third of the bowstring mass is added because just before the arrow leaves the bowstring the center of the bowstring is traveling at the arrow speed and its ends are essentially motionless. A simple integration yields the kinetic energy of the bowstring as

½ * 1/3 * bowstring mass * Varrow^2.

Why not use a model with *two* masses, M2 on the end of L21 and M3 on the tip? The answer is that it would greatly complicate the equations of motion since the kinetic energy of the mass on the tip involves both va1 *and* vx. The single mass model is sufficient to illustrate the transfer of energy from bow to arrow.

The results of the computation are the arrow speed, the static and dynamic force on the arrow and the tension in the bowstring, all versus the arrow position. The program is able to account for stretch in the bowstring but in this article we assume that the bowstring is inelastic.



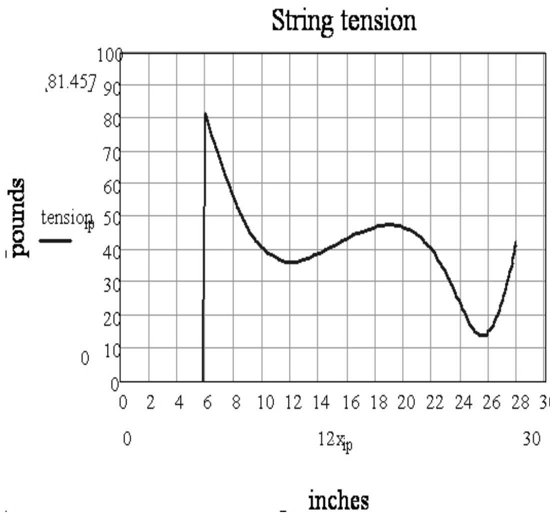


Fig.9. Static and dynamic force on the arrow.

Fig. 10. String tension versus arrow position.

It is interesting to compare static and dynamic forces versus arrow position as shown in Fig. 9. The static force is that required to draw the bowstring to a given position. The dynamic force is that which accelerates the arrow. The dynamic force is initially less than the static force because the bow limbs are also being accelerated. Later the bow limbs are decelerating and their kinetic energy is being transferred to the arrow and consequently the dynamic force on the arrow is greater than the static force. The string tension of Fig. 10 is greatest just before the arrow leaves the bowstring.